Power-law distribution as a result of asynchronous random switching between Malthus and Verhulst kinetics

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It is shown analytically that the flashing annihilation term of a Verhulst kinetic leads to the power-law distribution in the stationary state. For the frequency of switching slower than twice the free growth rate this provides the quasideterministic source of a Lévy noise at the macroscopic level.

I. INTRODUCTION

$1/f$ noise [1–6] and power-law distributions [6–9] are widely observed signatures of *anomalous* behavior [7,10]. Such phenomena are related to the scale invariance of considered systems and appear in different domains of natural and social sciences. Their universal character is partially explained on the statistical manner as a consequence of the central limit theorem for scaleless processes, where the Gaussian distribution is replaced by appropriate α-stable distribution [7,10]. This corresponds to the use of the Lévy noises to describe fluctuations or fractional dynamics at the level of Fokker-Planck equation [11,12]. Anomalous behavior can also be obtained for nonlinear systems driven by (conventional) multiplicative Gaussian white noise [8,6], particularly for the Lotka-Volterra (or Malthus-Verhulst) model [8]. In any case processes exhibiting fat-tailed distributions are subjected to large fluctuation, so within certain stochastic description it is difficult to distinguish to which extent such behavior is an intrinsic property of the kinetics and to which extent it is simply the reflection of the presence (and specific properties) of noise. The source of the possibly intense noise is known for thermodynamical critical systems. The search for explanation of scale invariant phenomena in *generic* systems is still unfinished [2–6,8]. Particularly, it is suggested in Ref. [2], see also Ref. [5], that the $1/f$ noise in a membrane channel is produced by random switching between both conducting states, rather than by inherent properties of ion transport. Similarly, the importance of activation-deactivation processes is addressed in Ref. [3].

In the present paper we are going to show analytically that the asynchronous switching between generic pure Malthusian and Verhulst-type kinetics leads to the Pareto distribution in the stationary state. The result seems interesting for few reasons. The logistic equation is used, which is the basic model of evolution for social sciences [13,14]. The power-law distribution was identified by Pareto to describe the social statistics, namely the distribution of (large) wealths. The model is *quasideterministic*, so the asymptotic power-law behavior has the kinetic origin only, coming as a result of balance between Malthusian growth and Verhulstian saturation. Finally, our result supports the general opinion of Ref. [2] about possible sources of anomalous behavior.

II. MODEL

Power-law behavior may be considered as an intermediate one between divergent and bounded kinetics. Let us note that the Malthus equation,

$$\dot{x}_t = ax, \tag{1}$$

where a is a positive difference between birth and death rate, leads to the exponential growth of population. Taking into account the Verhulst competition coefficient bx^μ, $\mu > 0$ (and equal unity for the "true" Verhulst model), depending on the actual size of population,

$$\dot{x}_t = ax - bx^{\mu+1}, \tag{2}$$

one obtains, instead of unbounded trajectories, the monotonic relaxation to the stationary value

$$x_{\mathrm{st}} = (a/b)^\nu, \quad \nu = 1/\mu, \tag{3}$$

with the relaxation time

$$T = 1/\mu a. \tag{4}$$

So, if the annihilation term of Eq. (2) is absent ($b=0$) the kinetic is divergent and as long as $b>0$ the kinetic is bounded. Thus one expects that the "flashing" annihilation term, temporarily switched on and off, may result in a power-law distribution for the asymptotic state. Let us simply assume a (Markovian) binary and asynchronous character of a switching process

$$b(t) = b[1 + \sigma I_t], \tag{5}$$

where $I_t = (-1)^{\mathcal{N}_t}$, where $\mathcal{N}_t$ is a Poisson process with parameter λ and $\sigma = +1$ or -1 with probability $1/2$. It means that the evolution consist of periods of active annihilation (with a coefficient $2b$) separated by periods of a free growth. The length of the periods is random with the average equal λ^{-1}.

Note that Eq. (2) is probably the most frequently studied one if stochastic effects on nonlinear kinetics are considered (see, e.g., [15] and references therein). The case of additive noise corresponds to an overdamped Brownian motion in some external potential and it is particularly important for nonequilibrium thermodynamics. The cases of multiplicative noise came both from population dynamics as the random growth rate or random carrying capacity models [13,14] and physical studies of a critical slowing down [16] and noise-induced transitions [17,15].

III. CRITICAL σ^2-DEPENDENCE OF MOMENTS

The Bernoulli equation (2) driven by dichotomous Markov process has been already considered [18,19]. Particu-

larly, in Ref. [19] the transient behavior of Eq. (2) with nonlinear coupling (5), $|\sigma| \le 1$, has been examined and in the limit $t \to \infty$ the following formula for the stationary moments

$$\langle x^s \rangle_{st} = x^s_{st2}F_1\left(\frac{s\nu}{2};\frac{s\nu+1}{2};1+\delta;\sigma^2\right), \quad (6)$$

where

$$\delta = \lambda/\mu a - 1/2, \quad (7)$$

was obtained. The value $\sigma^2 = 1$ is critical with respect to the convergence of the hypergeometric series (6) (*branch point* of the function) and for the critical exponent

$$1 + \delta - s\nu - 1/2 \equiv (\lambda/a - s)\nu \le 0, \quad (8)$$

the right-hand side of Eq. (6) diverges at $\sigma^2 = 1$. The condition (8) clearly shows the existence of fat-tail

$$P(x) \sim x^{-1-\lambda/a} \quad (9)$$

for large x.

IV. STATIONARY PROBABILITY DENSITY

The stationary probability density is formally given by the inverse Mellin transformation (with respect to s) of Eq. (6). The more convenient way of computation is to apply the so-called quadratic transformation of the hypergeometric function [20]

$$\langle x^s \rangle_{st} = x^s_{st}(1+\sigma)^{-\nu s}{}_2F_1\left(s\nu;\frac{1}{2}+\delta;1+2\delta;\frac{2\sigma}{1+\sigma}\right) \quad (10)$$

and then use of Euler's integral representation

$$\langle x^s \rangle_{st} = \frac{x^s_{st}(1+\sigma)^{-\nu s}}{B(\gamma,\gamma)}\int_0^1 dt\, t^{\gamma-1}(1-t)^{\gamma-1}\left[1-\frac{2\sigma t}{1+\sigma}\right]^{-\nu s}, \quad (11)$$

where $\gamma = 1/2 + \delta$ and B is the beta function. Introducing a new variable $\xi = [b(1+\sigma-2\sigma t)/a]^{-\nu}$ we obtain finally

$$\langle x^s \rangle_{st} = \int_{x_l}^{x_r} d\xi\, P(\xi)\xi^s, \quad (12)$$

where

$$P(x) = N^{-1}x^{-\mu-1}[\sigma^2 - (1-ax^{-\mu}/b)^2]^{\delta-1/2}, \quad (13)$$

and $x \in (x_l, x_r)$,

$$x_l = \left[\frac{a}{b(1+|\sigma|)}\right]^\nu, \quad x_r = \left[\frac{a}{b(1-|\sigma|)}\right]^\nu. \quad (14)$$

The normalization constant in Eq. (13) is equal to $N = B(\gamma,\gamma)(2|\sigma|)^{2\delta}b/(\mu a)$. Note that in the case of interest, $|\sigma| = 1$, the right boundary $x_r \to \infty$ and

$$P(x) \propto x^{-1-\lambda/a}(2-ax^{-\mu}/b)^{\delta-1/2}, \quad (15)$$

where $x \in [(a/2b)^\nu, \infty)$, in agreement with Eq. (9).

The stationary probability density (13) may also be obtained by considering the stochastic Liouville equation. In

fact for dichotomous noise the *explicit* expression is known for arbitrary "drift" and "diffusion" term in the Langevin equation [17,21]. Such approach is however useless if transient properties are considered. It is the special property of the Bernoulli equation that can be solved as the *explicit* functional depending on noise realization. In a number of cases the (functional) averaging can be done effectively [16,15] leading to time-dependent quantities.

V. NOISE-INDUCED TRANSITIONS

Let us remember that in the present work we consider Eqs. (2) and (5) with $|\sigma| = 1$ as a noiseless evolution consisting of two essentially different (and randomly switched) modes of deterministic kinetics. On the other hand the same model, especially for $|\sigma| \ll 1$, may be treated as a stochastic kinetic with nonlinearly coupled parametric noise, when the fluctuation of a coefficient b is described by a zero mean dichotomous color noise $\xi_t \equiv b\sigma I_t$, $\langle \xi_t \xi_0 \rangle = b^2\sigma^2 e^{-2\lambda t}$, of a correlation time τ and intensity D,

$$\tau = (2\lambda)^{-1}, \quad D = b^2\sigma^2\tau. \quad (16)$$

From this point of view it is worth analyzing the τ and D dependence within the context of so-called noise-induced phase transitions [17]. The case $\sigma^2 = 1$ corresponds to the critical line $b^{-2}D/\tau = 1$ in Fig. 1 and to the "true" transition, which is reflected by *singular* σ^2-dependence of observables, Eq. (6), and related power laws. Below this line the kinetic is bounded and consequently the support of stationary distribution is finite, Eq. (14). Above the line the system is *unstable* and x_t rapidly approaches infinity (after finite time). At the critical line the stationary states are the semiaxis distributions (lower curves in Fig. 1), Eq. (15), exhibiting fat tails with indices $\alpha = \lambda/a$. The dependence on the correlation time is also interesting. If the frequency of switching λ is small (long correlations), $\delta - 1/2 \equiv \lambda/\mu a - 1 < 0$, both boundaries x_l and x_r—which are the stationary values (3) of a deterministic kinetics (2) with coefficients $b(1+|\sigma|)$ and $b(1-|\sigma|)$ respectively—are attractive, the probability density (13) has a minimum and it is convex down. In contrast for λ greater than the relaxation rate μa, Eq. (4), the boundaries become repulsive. $P(x)$ is then unimodal. The $\lambda = \mu a$ [or $\tau = 1/(2\mu a)$] is a critical value with respect to the noise-induced transition in probability density.

VI. EXACT PARETO DISTRIBUTION

The case $\lambda = \mu a$, see the vertical line in Fig. 1, is exceptional for other reasons. The stationary distribution is then purely power, $P(x) \propto x^{-1-\mu}$, and neither diverges nor vanishes at the boundaries (14). And for $|\sigma| = 1$, at the *critical point* common for both lines, it becomes the exact Pareto distribution

$$P(x) = (a\mu/2b)x^{-1-\mu}, \quad x \ge (a/2b)^\nu. \quad (17)$$

Moreover, the case $\lambda = \mu a$ was the nontrivial one for which the time-dependent transient moments were found in the closed analytical form [19]
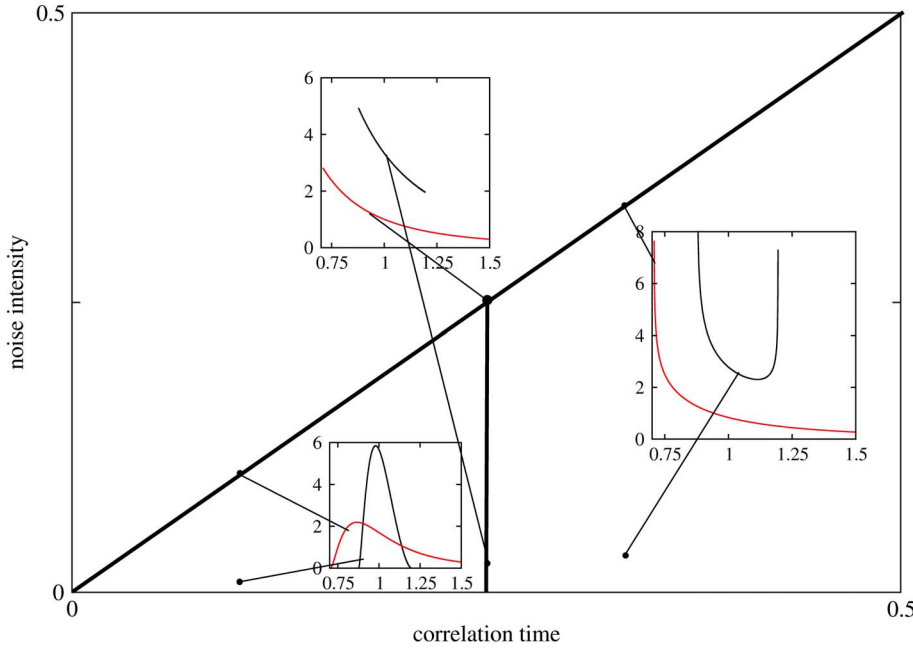
FIG. 1. (Color online) Phase diagram on (τ, D) plane, Eq. (16), for Stratonovich model $\mu = 2$ and dimensionless units $a = b = 1$. The vertical line $\tau = 0.25$ is related to noise-induced transition in the shape of probability density. The monotonic semiaxis distributions corresponding to $D = \tau > 0.25$ belong to the basin of attraction of Lévy α-stable distribution (with $\alpha = \lambda$). Insets: Plots of normalized $P(x)$ vs x for particular values of $(\delta, \sigma) \in \{1/4, 1/2, 2\} \times \{3/10, 1\}$. The location of (δ, σ) points on plane are marked by black dots, which correspond to coordinates $\tau = 1/10, 1/4, 1/3$ (or $\lambda = 5, 2, 3/2$, respectively) and $D = 0.009, 0.0225, 0.03$.

$$\langle x_t^s \rangle = e^{-\mu a t}[x^s(t;0) + x^s(t;2b)]/2 + \frac{a}{2b}$$

$$\times \begin{cases} \dfrac{1}{1 - s\nu}[x^{s-\mu}(t;2b) - x^{s-\mu}(t;0)], & \text{if } \mu \neq s, \\ s \ln[x(t;0)/x(t;2b)], & \text{if } \mu = s, \end{cases}$$

(18)

where

$$x(t;b) = [x_0^{-\mu}e^{-\mu a t} + b(1 - e^{-\mu a t})/a]^{-1/\mu} \qquad (19)$$

is the solution of Eq. (2) for a constant a and b parameters. Using the asymptotic forms $x(t;0) = x_0 e^{at}$ and $x(t;2b) \rightarrow (a/2b)^\nu$, one concludes that the higher order, $s > \mu$, transient moments diverge exponentially

$$\langle x_t^s \rangle \propto e^{(s-\mu)a t} \qquad (20)$$

and for the marginal case $s = \mu (= \lambda/a)$

$$\langle x_t^\mu \rangle \rightarrow \text{const} + \frac{a^2 \mu}{2b} t \qquad (21)$$

grows linearly for long times. Note that according to Eq. (15) or Eq. (9) the value of the ratio $\alpha = \lambda/a$ (frequency of switching by the rate of a free growth) specifies the order of the lowest divergent moment. In the "double-critical" case of exact Pareto distribution (17) it is simultaneously equal to the value of a "degree of nonlinearity" μ. Thus, for the most important true Verhulst or so-called Stratonovich (with $\mu = 2$, corresponding to the quartic potential) model, it means that it is the mean value or the variance, respectively, that grows linearly with time, Eq. (21).

VII. CONCLUSIONS

The anomalous systems exhibit large fluctuations which are difficult to explain dynamically within standard theories,

for instance, as a result of an additive Gaussian diffusion. Thus such a behavior is frequently joined with the presence of a "large" driving noise, e.g., a Lévy noise or a multiplicative Gaussian noise. On the other hand it was suggested that certain aspects of such behavior are rather related to some hierarchical structure of kinetic processes at the deterministic level [2,3,5]. This idea was exploited in the present work for exactly solvable generic model (2) and (5). The general behavior of the system depends on two things: On the mutual relation between λ and a (or μa) and on the value of σ^2, namely whether $\sigma^2 < 1$ or $\sigma^2 = 1$.

Let $\sigma^2 < 1$ (i.e., below the critical line in Fig. 1). If frequency λ is smaller than the rate of relaxation processes μa the x_t "hardly" follows $b(t)$, Eq. (5), switching between some values located close to x_l or x_r, Eq. (14), respectively (right part of plot). Successive ("rare" and "short") moves up and down are associated with values $b(1 - |\sigma|)$ and $b(1 + |\sigma|)$ in Eq. (2), respectively, and in principle look like different deterministic processes. In the opposite case (left part of plot), $\lambda > \mu a$, the evolution looks more like the Verhulst kinetics with the averaged value $b = \overline{b(t)}$ of annihilation coefficient. The distribution of x is unimodal, however still with a rather wide maximum, located somewhere in a middle of the support. The value $\lambda = \mu a$ is critical for so-called noise induced transition. This transition is formally controlled by correlation time of the noise.

The case of $\sigma^2 = 1$ is of particular interest for the present work and may be formally considered as the "true" transition, which is indicated by singular dependence of moments and related power laws. This transition is controlled by the noise intensity. The asymptotic distribution of x is given by Eq. (15). $P(x)$ has a fat-tail with the index $\alpha = \lambda/a$. The distributions with $\alpha < 2$ belong—via central limit theorem—to the basin of attraction of appropriate Lévy α-stable distribution. Thus the kinetic (2) and (5) with randomly flashing annihilation term may be treated as a source of a Lévy noise

at the macroscopic level, if the frequency of switching is small enough, $\lambda < 2a$, or, equivalently, if the correlation time is sufficiently long $\tau > (4a)^{-1}$. For the Stratonovich model, which is presented in Fig. 1, this corresponds precisely to positions on the line right from the critical point [with $\tau = (2\mu a)^{-1}$]. If the frequency is large, $\lambda > 2a$, or the correlations are short, $P(x)$ still has a fat tail; however, it corresponds to the Gaussian distribution via central limit theorem.

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